Radiative Cooling of a Layer with Nonuniform Velocity: A Separable Solution

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Analyzed here is the cooling of a flowing plane layer filled with hot particles or liquid droplets that emit, absorb, and scatter radiation. The velocity distribution is nonuniform across the layer and the specified shape of this distribution affects the layer temperature distribution and emittance. The velocity distribution determined is one that will cause the layer to cool at a uniform temperature. Thus, if the layer initially has a uniform temperature, it will retain the same emittance throughout its length. A separable solution is found that applies in a "fully developed" region following an initial cooling length. In this developed region, the solution shows that there is a constant emittance based on the local heat loss and local bulk mean layer temperature. This emittance is a function of the velocity distribution, optical thickness, and scattering albedo.

Nomenclature

A	= amplitude in parabolic velocity distribution
a	= absorption coefficient of absorbing-scattering layer
c_p	= specific heat of droplet-filled layer
D	= thickness of absorbing-scattering layer
E_1,E_2,E_3	= exponential integral functions, $E_n(x) = \int_0^1 \mu^{n-2} \exp(-x/\mu)\mathrm{d}\mu$
$F(X)$	= function of X in temperature profile
$G(X)$	= function of X in profile of $\tilde{I}^{1/4}$
I	= source function in radiating layer, $\tilde{I} = \pi I/\sigma T_m^4(0)$
q	= local heat loss per unit area and time from one side of layer
q_r	= radiative heat flow per unit area and time
T	= absolute temperature, $\tilde{T} = T/T_m(0)$
T_e	= environment temperature
$T_m(Z)$	= bulk mean temperature as a function of length
$u(x)$	= velocity distribution within layer, $U = u/u_m$
u_m	= integrated mean velocity of layer
x,z	= coordinates across and along layer
X	= dimensionless coordinate, $x/D = \kappa/\kappa_D$
X^*	= dummy variable of integration
Z	= dimensionless coordinate, $4\sigma T_m^3(0)z/u_m\rho c_p D$
ϵ_{fd}	= emittance of layer in fully developed region
θ	= function of Z in profiles of $\tilde{T}$ and $\tilde{I}^{1/4}$
κ	= optical coordinate $(a+\sigma_s)x$; κ_D optical thickness of layer $(a+\sigma_s)D$
κ^*	= dummy variable of integration
ρ	= density of droplet-filled layer
σ	= Stefan-Boltzmann constant
σ_s	= scattering coefficient of layer
Ω	= albedo for scattering, $\sigma_s/(a+\sigma_s)$

Introduction

A DEVICE for waste heat dissipation in space applications has been proposed that will utilize regions filled with hot liquid or solid droplet streams projected through space.[1,2] The droplet streams cool by radiative loss and then pass into a collection device to be reused. The engineering aspects and optimization of the device are discussed in detail in Refs. 1 and 2; it is the radiative heat-transfer behavior that is of interest here. The droplet-filled region is analyzed in the form of a plane layer. If a layer is generated at uniform velocity and at uniform temperature, the subsequent cooling as it travels through space will cause a temperature distribution to develop across the layer. In a transient cooling analysis of a layer having a uniform velocity distribution,[3] it was found that the layer emittance based on the local heat loss and local mean temperature achieved a constant value as the layer continued to cool. The region where the emittance becomes constant is called here the "fully developed" cooling region. The evolution from the initial temperature distribution to the fully developed distribution was shown by the transient solution in Ref. 3.

The present analysis will show that a fully developed cooling region also occurs when the velocity distribution across the layer is nonuniform. The shape of the velocity distribution provides a degree of freedom that can be used to influence the layer temperature distribution and emittance. As an interesting and useful special case, the velocity distribution is found that will cause the layer to retain a uniform temperature distribution throughout the entire cooling process. This velocity distribution is a function of the optical thickness of the layer and its scattering albedo. When the scattering albedo is zero (absorption only), an analytical solution is found for the velocity distribution that will provide uniform temperature cooling.

This analysis is a continuation of the study in Ref. 3 of the transient radiative cooling in a scattering plane layer, relative to the proposed liquid drop radiator.[1,2] Some general background information on the analysis and nature of transient radiative cooling is given in Refs. 4 and 5. Some additional analyses on transient radiative heat transfer in plane layers are described in Refs. 6–8. These references have not treated the cooling situation studied here of a droplet layer moving at nonuniform velocity.

Analysis

The layer of heated droplets moving through the vacuum of space is shown in Fig. 1. Radiative cooling is occurring to an environment at zero temperature, $T_e = 0$. The droplet layer is projected through space with a nonuniform velocity that is a function of position across the layer. The velocity does not change with the distance traveled z. It will be shown that a quasi-equilibrium condition is reached wherein the emittance of the layer becomes constant; this occurs in what will be

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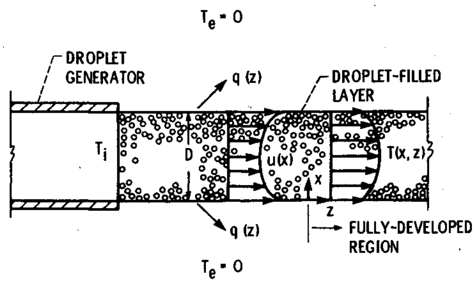


Fig. 1 Geometry of radiating-scattering layer filled with hot droplets and having a nonuniform velocity.

called the "fully developed" cooling region. The emittance is a function of the velocity distribution; hence, the velocity distribution can be adjusted to raise or lower the emittance.

Separation of Variables Solution

The convective and radiative energy equation for the droplet layer is

$$\frac{\rho c_p}{a+\sigma_s} u(\kappa) \frac{\partial T(\kappa,z)}{\partial z} = -\frac{\partial q_r}{\partial \kappa} \tag{1}$$

where the derivative of q_r in the z direction has been neglected relative to that in the transverse direction. Since the radiator is operating in the vacuum of space, heat conduction between droplets is neglected. Substituting $\partial q_r/\partial \kappa$ for a plane layer[5] gives

$$\frac{\rho c_p}{a+\sigma_s} \frac{\partial T(\kappa,z)}{\partial z} = \frac{2\pi}{u(\kappa)} \times \left[\int_0^{\kappa_D} I(\kappa^*,z)E_1(|\kappa-\kappa^*|)\mathrm{d}\kappa^* - 2I(\kappa,z)\right] \tag{2a}$$

where the source function for scattering (assumed isotropic) in a gray layer is

$$I(\kappa,z) = (1-\Omega)\frac{\sigma T^4(\kappa,z)}{\pi} + \frac{\Omega}{2}\int_0^{\kappa_D} I(\kappa^*,z)E_1(|\kappa-\kappa^*|)\mathrm{d}\kappa^* \tag{2b}$$

Using the dimensionless quantities in the Nomenclature, Eqs. (2a) and (2b) become

$$\frac{\partial \tilde{T}}{\partial Z} = \frac{\kappa_D}{U(X)}\left[\frac{\kappa_D}{2}\int_0^1 \tilde{I}(X^*,Z)E_1(\kappa_D|X-X^*|)\mathrm{d}X^* - \tilde{I}(X,Z)\right] \tag{3a}$$

$$\tilde{I}(X,Z) = (1-\Omega)\tilde{T}^4(X,Z) + \frac{\Omega\kappa_D}{2}\int_0^1 \tilde{I}(X^*,Z)E_1(\kappa_D|X-X^*|)\mathrm{d}X^* \tag{3b}$$

A separable solution for Eqs. (3a) and (3b) is now tried in the form

$$\tilde{T}(X,Z) = \theta(Z)F(X), \qquad F(0)=1 \tag{4a}$$

$$\tilde{I}^{1/4}(X,Z) = \theta(Z)G(X) \tag{4b}$$

where $F(0)$ has been arbitrarily set equal to unity. The separable forms are inserted into Eqs. (3a) and (3b) to yield,

$$\frac{1}{\theta^4(Z)}\frac{\mathrm{d}\theta}{\mathrm{d}Z} = \frac{\kappa_D}{F(X)U(X)} \times \left[\frac{\kappa_D}{2}\int_0^1 G^4(X^*)E_1(\kappa_D|X-X^*|)\mathrm{d}X^* - G^4(X)\right] \tag{5a}$$

$$G^4(X) = (1-\Omega)F^4(X) + \frac{\Omega\kappa_D}{2}\int_0^1 G^4(X^*)E_1(\kappa_D|X-X^*|)\mathrm{d}X^* \tag{5b}$$

In Eq. (5a) the left side contains only functions of Z and the right side only functions of X. Hence, the two sides must equal the same constant. The right side can then be equated to its evaluation at $X=0$ to yield an integral equation involving G and F but not θ,

$$F(X) = \frac{U(0)}{U(X)} \frac{\frac{\kappa_D}{2}\int_0^1 G^4(X^*)E_1(\kappa_D|X-X^*|)\mathrm{d}X^* - G^4(X)}{\frac{\kappa_D}{2}\int_0^1 G^4(X^*)E_1(\kappa_D|0-X^*|)\mathrm{d}X^* - G^4(0)} \tag{6}$$

Equation (6) is combined with Eq. (5b) to eliminate the integrals of G^4. This yields

$$F^4(X) = G^4(X) + F(X)\frac{U(X)}{U(0)}[1-G^4(0)] \tag{7}$$

With $U(X)$ given, Eqs. (7) and (5b) were solved numerically for $F(X)$ and $G(X)$. For the special case when there is no scattering ($\Omega=0$), $G(X)\to F(X)$ (see Ref. 5) and Eq. (6) yields the following integral equation for $F(X)$:

$$F^4(X) = \frac{U(X)}{U(0)}F(X)\left[1-\frac{\kappa_D}{2}\int_0^1 F^4(X^*)E_1(\kappa_D|0-X^*|)\mathrm{d}X^*\right] + \frac{\kappa_D}{2}\int_0^1 F^4(X^*)E_1(\kappa_D|X-X^*|)\mathrm{d}X^* \tag{8}$$

After $F(X)$ is found, the temperature distribution in the "fully developed" cooling region is obtained by first integrating Eq. (4a) to obtain the mean temperature,

$$\tilde{T}_m = \frac{T_m(Z)}{T_m(0)} = \theta(Z)\int_0^1 U(X)F(X)\mathrm{d}X \tag{9}$$

Then, after dividing Eq. (4a) by Eq. (9), the normalized temperature distribution is

$$\frac{T(X,Z)}{T_m(Z)} = \frac{F(X)}{\int_0^1 U(X)F(X)\mathrm{d}X} \tag{10}$$

Relations for Layer Emittance

To obtain the constant layer emittance in the "fully developed" cooling region, an energy balance for a length dz of the layer yields

$$2\epsilon_{fd}\sigma T_m^4 = -\rho c_p \frac{\mathrm{d}}{\mathrm{d}z}\int_0^D u(x)T(x)\mathrm{d}x = -\rho c_p D u_m \frac{\mathrm{d}T_m(z)}{\mathrm{d}z}$$

In dimensionless form this equation is,

$$\frac{\epsilon_{fd}}{2}\tilde{T}_m^4(Z) = -\frac{d\tilde{T}_m(Z)}{dZ} \tag{11}$$

By substituting Eq. (9) this becomes,

$$\frac{\epsilon_{fd}}{2}\left[\int_0^1 U(X)F(X)dX\right]^3 = -\frac{1}{\theta^4(Z)}\frac{d\theta}{dZ} \tag{12}$$

Then by use of Eq. (5a), evaluated for convenience at $X=0$, an expression for the layer emittance is

$$\epsilon_{fd} = \frac{2}{\left[\int_0^1 U(X)F(X)dX\right]^3}\frac{\kappa_D}{U(0)}\left[G^4(0) - \frac{\kappa_D}{2}\int_0^1 G^4(X^*)E_1(\kappa_D|0-X^*|)dX^*\right] \tag{13a}$$

Another form is obtained by use of Eq. (5b) to eliminate the integral of G^4,

$$\epsilon_{fd} = \frac{2}{\left[\int_0^1 U(X)F(X)dX\right]^3}\frac{\kappa_D}{U(0)}\frac{1-\Omega}{\Omega}[1-G^4(0)] \tag{13b}$$

When $\Omega=0$, Eq. (13a) is used with G replaced by F calculated from Eq. (8).

Velocity Distribution to Obtain Constant-Temperature Layer

A useful inverse design situation is to determine how to provide the appropriate velocity distribution (a function of X, but not of Z) that will yield a uniform local temperature distribution across the radiating layer. Then the layer, which is initiated at uniform temperature, will remain at uniform temperature for all Z and hence the layer emittance will be constant throughout the entire cooling process. For a uniform temperature, $F(X)=1$, so from Eq. (7)

$$\frac{U(X)}{U(0)} = \frac{u(x)}{u(0)} = \frac{1-G^4(X)}{1-G^4(0)} \tag{14}$$

By integrating over X to obtain $u_m/u(0)$, the velocity distribution can be expressed as

$$\frac{u(x)}{u_m} = \frac{1-G^4(X)}{\int_0^1 [1-G^4(X)]dX} \tag{15}$$

$G(X)$ is obtained by solving Eq. (5b) with $F(X)=1$,

$$G^4(X) = (1-\Omega) + \frac{\Omega\kappa_D}{2}\int_0^1 G^4(X^*)E_1(\kappa_D|X-X^*|)dX^* \tag{16}$$

For the special case when there is no scattering ($\Omega=0$), Eq. (8) gives, using $F(X)=1$,

$$\frac{U(X)}{U(0)} = \frac{\frac{\kappa_D}{2}\int_0^1 E_1(\kappa_D|X-X^*|)dX^* - 1}{\frac{\kappa_D}{2}\int_0^1 E_1(\kappa_D|0-X^*|)dX^* - 1}$$

$$= \frac{E_2(\kappa_D X) + E_2[\kappa_D(1-X)]}{1+E_2(\kappa_D)} \tag{17}$$

Then

$$\frac{u_m}{u(0)} = \frac{1}{U(0)} = \int_0^1 \frac{U(X)}{U(0)}dX = \frac{1}{\kappa_D}\frac{1-2E_3(\kappa_D)}{1+E_2(\kappa_D)} \tag{18}$$

so that from Eqs. (17) and (18) the required velocity distribution is

$$\frac{u(x)}{u_m} = \frac{\kappa_D\{E_2(\kappa_D X) + E_2[\kappa_D(1-X)]\}}{1-2E_3(\kappa_D)} \tag{19}$$

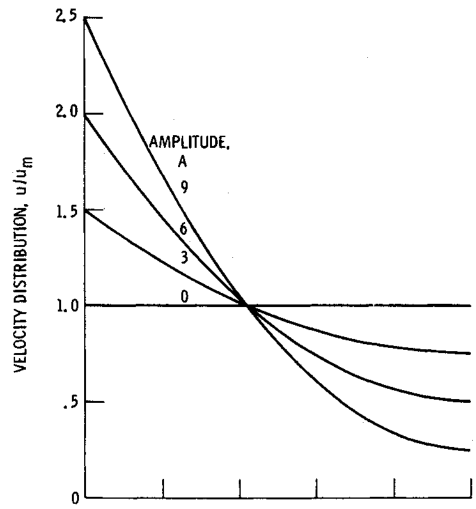
For $\Omega=0$ so that $G \to F$, ϵ_{fd} is found from Eq. (13a) with the use of Eq. (18),

$$\epsilon_{fd} = \frac{2\kappa_D}{U(0)}\left[1-\frac{\kappa_D}{2}\int_0^1 E_1(\kappa_D|0-X^*|)dX^*\right] = 1-2E_3(\kappa_D) \tag{20}$$

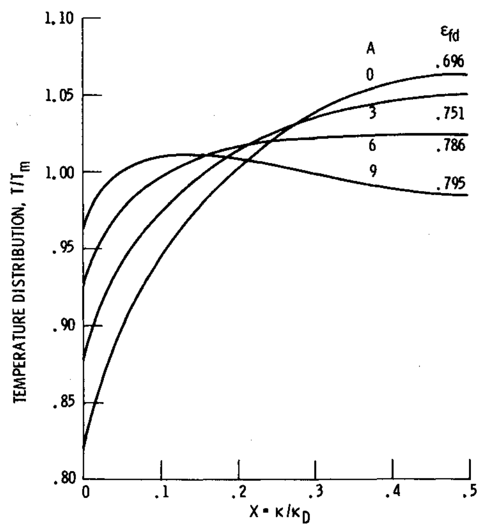
This is the known emittance relation for an absorbing layer at uniform temperature.[5]

Velocity Distribution for Typical Results

Some typical results will be given to demonstrate how an imposed velocity distribution affects the emittance and temperature distribution of the layer. For this purpose, a

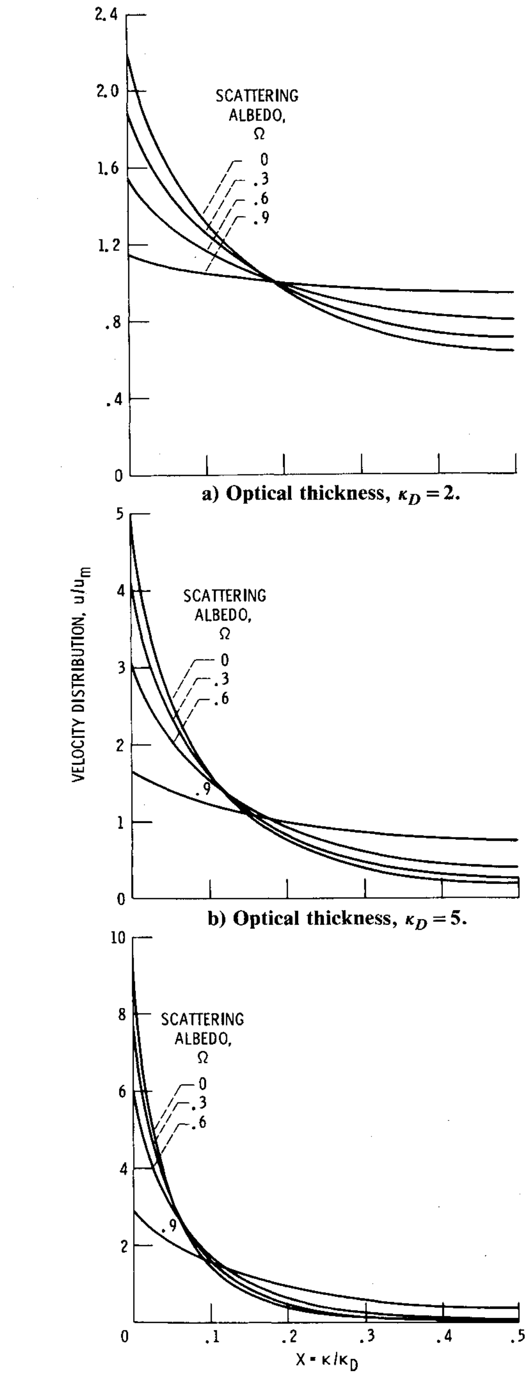


a) Velocity distributions for various A.



b) Temperature distributions.

Fig. 2 Effect of velocity distribution on fully developed temperature distribution for layer with $\kappa_D=5$ and $\Omega=0.6$.



a) Optical thickness, $\kappa_D = 2$.

b) Optical thickness, $\kappa_D = 5$.

c) Optical thickness, $\kappa_D = 10$.

Fig. 3 Velocity distributions as a function of κ_D and Ω that will cause the layer to cool at uniform temperature.

parabolic velocity distribution was used having an amplitude A,

$$U(X) = \frac{u(x)}{u_m} = 1 + A\left(\frac{1}{6} - X + X^2\right) \tag{21}$$

Mean Temperature Along Length of Layer

The variation of mean temperature T_m with length Z is found by integrating Eq. (11). Let $Z = 0$ be at any location in the "fully developed" region shown in Fig. 1. Since the dimensionless $\tilde{T}_m(Z=0) = 1$ and ϵ_{fd} is a constant, the integration from $Z = 0$ to Z yields

$$\frac{T_m(Z)}{T_m(0)} = \frac{1}{[1 + (3/2)\epsilon_{fd} Z]^{1/3}} \tag{22}$$

Table 1 Values of layer emittance when velocity distribution provides cooling at uniform temperature

Optical thickness κ_D	Scattering albedo Ω: 0	0.3	0.6	0.9
2	0.940	0.846	0.681	0.297
5	0.998	0.924	0.798	0.470
10	1.000	0.933	0.808	0.518

When the velocity distribution has the specific form that will yield cooling with a uniform temperature distribution, there is no cooling length required to achieve the fully developed temperature distribution, since the layer leaves the droplet generator at uniform temperature. Equation (22) will then apply for the entire cooling length of the layer starting from the exit face of the droplet generator.

Numerical Solution

For layers with $\Omega > 0$, $F(X)$ and $G(X)$ were obtained numerically by solving Eqs. (5b) and (7). Trial functions for $F^4(X)$ and $G^4(X)$, based on the shapes obtained in previous work,[3] were substituted into the right side of Eq. (5b); this equation was solved for $G^4(X)$ by iteration, keeping $F^4(X)$ fixed. At the end of each iteration, the difference between the right side of Eq. (5b) and the trial $G^4(X)$ was multiplied by an acceleration factor of 1.2 and the result added to the trial $G^4(X)$; this gave a new $G^4(X)$ with which to continue the iteration. After convergence to within about 10^{-4}, $G^4(X)$ and $F(X)$ were used on the right side of Eq. (7) to obtain $F^4(X)$ on the left side. Then an acceleration factor, usually 1.5, was applied to this $F^4(X)$ to obtain a new $F^4(X)$ with which to repeat the iteration of Eq. (5b) for obtaining a new $G^4(X)$.

The $F^4(X)$ from Eq. (8) and the $G^4(X)$ from Eq. (16) were obtained in a similar iterative manner. The final results for temperature distribution, emittance, and velocity distribution were then readily evaluated from Eqs. (10), (13), and (15).

The numerical solution requires an accurate integration technique. A library subroutine for Gaussian integration was used and the solutions generally required a few minutes or less on an IBM 370 computer. The integration kernel in Eq. (5b) has a singularity since $E_1(0) = \infty$; hence, special attention is needed as X^* approaches X. F and G were held constant for a very small region adjacent to the singularity and the integration over this region was performed analytically. This was easily done since the integral of E_1 is $-E_2$ and $E_2(0) = 1$. For small arguments, E_2 was calculated from its known series expansion. The size of the small region was decreased to a range that did not affect the computed results.

Results and Discussion

The drawing in Fig. 1 shows a droplet layer having a nonuniform velocity distribution with its largest velocity at the outer edge of the layer. This distribution will provide increased convective energy transport into the outer regions of the layer and will prevent the outer regions from cooling as rapidly as for a layer with uniform velocity. This will help maintain a high layer emittance. To illustrate this effect, the velocity distribution in Eq. (21) was used in Eq. (7) with the four A values shown in Fig. 2a. The resulting temperature distributions for $\kappa_D = 5$ and $\Omega = 0.6$ are shown in Fig. 2b as obtained with the aid of Eq. (10). These are the distributions that exist throughout the fully developed region within which the emittance ϵ_{fd} is a constant. The ν_{fd} values are also included in the figure. As the velocity is increased within the outer regions of the layer, the temperature profile becomes more uniform and the ϵ_{fd} value increases.

Now we turn our attention to the shape of the velocity distribution required to keep the temperature distribution uniform for all z locations. In this instance, the heat-transfer behavior will be fully developed for the entire flow, starting from the exit face of the droplet generator where the droplet layer is initiated at uniform temperature. The emittance will be that for a layer at uniform temperature and will be a function of only κ_D and Ω. The required velocity distributions are calculated from Eqs. (15) and (16) for $\Omega > 0$. For a layer with no scattering, $\Omega = 0$, the analytical solution is given by Eq. (19).

The three parts of Fig. 3 are for three different κ_D values: 2, 5, and 10. Since the temperature distribution is uniform for all cases, the ϵ_{fd} depend only on κ_D and Ω and are given in Table 1 as obtained from Eqs. (13a) and (20). For an optically thick region, the outer portions of the layer cool much more readily than the inner portions; hence, there is a tendency for large temperature nonuniformities to develop. Thus, maintaining a uniform temperature across the layer requires a larger nonuniformity in the velocity distribution as the optical thickness is increased. This trend is evident on the parts of Fig. 3. As the scattering albedo increases, the increased reflection of energy between the droplets tends to smooth the layer temperature distribution. Hence, the magnitude of the nonuniformity in velocity, required to maintain a uniform temperature, decreases as the scattering albedo Ω is increased.

Conclusions

A separable solution has been found for the temperature and source function distributions during radiative cooling of a flowing plane layer with a nonuniform velocity distribution. The solution shows that there is a "fully developed" cooling region in which the layer emittance is a constant that depends only on the optical thickness and scattering albedo. This emittance is based on the instantaneous bulk mean temperature and the heat loss from one side of the layer. The shape of the velocity distribution in the layer can be used to control the temperature distribution and emittance. Found is the velocity distribution that will cause the layer to remain at uniform temperature throughout its entire cooling period.

References

[1]Mattick, A.T. and Hertzberg, A., "Liquid Droplet Radiators for Heat Rejection in Space," *Journal of Energy*, Vol. 5, Nov.-Dec. 1981, pp. 387-393.

[2]Presler, A.F., Coles, C.E., Diem-Kirsop, P.S., and White, K.A., "Liquid Droplet Radiator Program at the NASA Lewis Research Center," ASME Paper 86-HT-15, June 1986.

[3]Siegel, R., "Transient Radiative Cooling of a Droplet-Filled Layer," *Journal of Heat Transfer*, Vol. 109, Feb. 1987, pp. 159-164.

[4]Viskanta, R. and Anderson, E.E., "Heat Transfer in Semitransparent Solids," *Advances in Heat Transfer*, Vol. 11, edited by T.F. Irvine Jr. and J.P. Hartnett, Academic Press, New York, 1975, pp. 317-441.

[5]Siegel, R. and Howell, J.R., *Thermal Radiation Heat Transfer*, 2nd ed., Hemisphere Publishing Co., Washington, DC, 1981.

[6]Viskanta, R. and Bathla, P.S., "Unsteady Energy Transfer in a Layer of Gray Gas by Thermal Radiation," *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 18, No. 3, May 1967, pp. 353-367.

[7]Bathla, P.S. and Viskanta, R., "Effect of Surroundings on the Transient Energy Transfer in a Layer of Radiating Gas," *Applied Scientific Research*, Vol. 19, July 1968, pp. 182-197.

[8]Kubo, S., "Unsteady Radiative Heat Transfer in a Scattering-Dominant Medium," *AIAA Journal*, Vol. 22, Dec. 1984, pp. 1804-1809.